

ON A DIFFERENTIAL INCLUSION RELATED TO THE
BORN–INFELD EQUATIONS*STEFAN MÜLLER[†] AND MARIAPIA PALOMBARO[‡]

Abstract. We study a partial differential relation that arises in the context of the Born–Infeld equations (an extension of Maxwell’s equations) by using Gromov’s method of convex integration in the setting of divergence-free fields.

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1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $K \subset \mathbb{M}^{m \times n}$ be a set of $m \times n$ real matrices. We study the problem of whether there exist solutions to the differential inclusion

$$(1.1) \quad \begin{cases} \operatorname{Div} V = 0 & \text{in } \mathcal{D}'(\Omega; \mathbb{R}^m), \\ V \in K & \text{a.e. in } \Omega, \\ \int_{\Omega} V = F \end{cases}$$

for some given $F \in \mathbb{M}^{m \times n}$. Our interest in this question arises, in particular, from applications to the study of the Born–Infeld equations. In fact, we will consider a special case of (1.1), when $m = 2$, $n = 3$, and the set K is related to the so-called Born–Infeld manifold. Further applications of solenoidal differential inclusions can be found in the study of composite materials, as well as linear elasticity and fluid mechanics (see, e.g., [6, 7, 9, 16, 17]). More generally, problem (1.1) falls into the framework of \mathcal{A} -quasi-convexity, where the differential constraint on the function V is replaced by more general ones (see, e.g., [8] and [18] for related issues).

Our approach to (1.1) is based on studying the method of convex integration in the div-free setting. Convex integration has been introduced and developed by Gromov to solve partial differential relations, in particular in connection with geometric problems. An important problem is to find gradient fields that take values in a prescribed set of matrices. This can be written as the partial differential relation

$$(1.2) \quad \nabla u \in K.$$

We refer to Gromov’s treatise [10] for a detailed exposition and further references concerning the existence of C^1 solutions. Gromov only very briefly discusses the existence of Lipschitz solutions (see [10, p. 218]) and a more detailed theory of Lipschitz solutions has been developed in a number of contributions, including [14, 2, 3, 21, 12, 13]

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[†]Hausdorff Center for Mathematics and Institute for Applied Mathematics, Universität Bonn, 53115 Bonn, Germany (sm@hcm.uni-bonn.de).

[‡]Department of Information Engineering, Computer Science and Mathematics, University of L’Aquila, 67100 L’Aquila, Italy (mariapia.palombaro@univaq.it).

and has lead to a number of new results, e.g., in the study of solid-solid phase-transitions, counterexamples to the regularity of elliptic systems [15, 22], and mathematical origami [5]. The partial differential relation (1.2) corresponds (locally) to the constraint $\operatorname{curl} v = 0$. In the spirit of Tartar's work [24] it is natural to consider also constraints $\mathcal{A}v = 0$, where \mathcal{A} is a general first order differential operator with constant coefficients. We deal with the case $\operatorname{div} V = 0$, where V is matrix-valued and the divergence is taken rowwise. The divergence constraint has already been considered elsewhere in the context of convex integration, e.g., in [4] in the context of general closed differential forms¹ and in [6, 7] in the context of the Euler equations (see also [23] and references therein). In section 3 we give a brief self-contained description of convex integration with the constraint $\operatorname{div} V = 0$ since we will use exactly the same strategy for the application to the Born–Infeld equation.

More precisely we show in Theorem 3.8 that problem (1.1) admits a solution whenever K can be “approximated” in the sense of Definition 3.7 and F lies in the interior of some appropriate hull of K .

In section 4 we specialize the results obtained in section 3 to the case of a partial differential relation arising in connection with the Born–Infeld equations. Let us briefly introduce the problem. The Born–Infeld system is a nonlinear version of Maxwell's equations which can be written as a set of partial differential constraints

$$(1.3) \quad \partial_t D + \operatorname{curl} \left(\frac{-B + D \wedge P}{h} \right) = \partial_t B + \operatorname{curl} \left(\frac{D + B \wedge P}{h} \right) = 0,$$

$$(1.4) \quad \operatorname{div} D = \operatorname{div} B = 0,$$

combined with the pointwise relation

$$(1.5) \quad P = D \wedge B, \quad h = \sqrt{1 + |B|^2 + |D|^2 + |P|^2}.$$

Here $D, B, P : \Omega \times [0, T] \subset \mathbb{R}^3 \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$, and $h : \Omega \times [0, T] \subset \mathbb{R}^3 \times \mathbb{R}^+ \rightarrow \mathbb{R}$. Note that (1.3) implies that $\partial_t \operatorname{div} D = \partial_t \operatorname{div} B = 0$. Thus if (1.4) holds at time $t = 0$ it holds for all times.

The relations (1.5) define a six-dimensional manifold in \mathbb{R}^{10} , which we call the BI-manifold and denote by \mathcal{M} . We refer to Brenier [1] for the mathematical analysis and many further references on the Born–Infeld equations (1.3). Here we only give a brief account of those arguments of [1] which give rise to the question addressed in this paper. The starting point is to observe that if (D, B) are smooth solutions of (1.3) and if P and h are given by (1.5), then they satisfy the additional conservation laws

$$(1.6) \quad \partial_t h + \operatorname{div} P = 0,$$

$$(1.7) \quad \partial_t P + \operatorname{Div} \left(\frac{P \otimes P - B \otimes B - D \otimes D}{h} \right) = \nabla \left(\frac{1}{h} \right).$$

This suggests lifting the 6×6 system (1.3) to a 10×10 system of conservation laws by adding (1.6), (1.7), regardless of the condition (1.5). More precisely, one regards P and h as additional unknowns and considers the augmented system (1.3), (1.6),

¹The setting in [4] is both more general and more restrictive than our setting. First the authors consider the relation $d\omega \in E$, where $\omega \in W^{1,\infty}(\Omega, \Lambda^k)$ is a general k -form on $\Omega \subset \mathbb{R}^n$ (our setting corresponds to $k = n - 2$), and second they allow E to be contained in a lower-dimensional subspace. On the other hand their treatment does not directly cover the case $\operatorname{div} V = 0$ if $V \in W^{1,\infty}(\Omega, \mathbb{M}^{m \times n})$ and $m \geq 2$.

(1.7). This system enjoys remarkable properties which allow for an easier analysis than the original system (1.3) (see [1] for more precise details). Of course, among all solutions of the augmented system, only those with initial conditions valued in the BI-manifold genuinely correspond to the original system (1.3). A natural question is which initial conditions can be weakly approximated by initial conditions valued in the BI-manifold \mathcal{M} . Brenier shows that the convex hull of the six-dimensional set \mathcal{M} contains an open set in \mathbb{R}^{10} and then states without proof, “From this result, we infer that, through weak completion, we may consider, for the ABI system, all kinds of initial condition with full dimensionality, where the ‘fluid variables’ (h, P) are clearly distinct from the ‘electromagnetic’ variables” [1, p. 73].

Here we provide a proof of the statement that all vectors F in the interior of the convex hull of \mathcal{M} can arise as initial conditions through completion. A soft version of this statement is that given $F \in \text{Int}(\mathcal{M}^c)$ there exists $\{V_j\} = \{(D_j, B_j, P_j, h_j)\} \subset L^p(\Omega; \mathbb{R}^{10})$ such that

$$(1.8) \quad V_j \rightharpoonup F \quad \text{weakly in } L^p$$

and

$$(1.9) \quad \text{div} D_j = \text{div} B_j = 0 \quad \text{and} \quad \text{dist}(V_j, \mathcal{M}) \rightarrow 0 \quad \text{in measure}$$

or

$$\text{div} D_j \rightarrow 0, \text{div} B_j \rightarrow 0 \quad \text{strongly in } W^{-1,p'} \quad \text{and} \quad V_j \in \mathcal{M} \text{ a.e.}$$

This is proved in section 4 and is essentially a consequence of Tartar’s approach; see [24] and [8].

In section 4.2 we prove the following stronger result, which shows that there exists an approximating sequence which satisfies both constraints $\text{div} B_j = \text{div} D_j = 0$ and $V_j \in \mathcal{M}$ exactly. Since it requires almost no extra work we allow piecewise constant functions F rather than just constant F .

Here and in the following we say that a function $f : \Omega \rightarrow \mathbb{R}^m$ is *piecewise constant* if there exist (finitely or countably many) mutually disjoint open sets Ω_i with Lipschitz boundary such that

$$(1.10) \quad f|_{\Omega_i} \text{ is constant} \quad \text{and} \quad |\Omega \setminus \bigcup_i \Omega_i| = 0,$$

where $|E|$ denotes the Lebesgue measure of E . Similarly we say that f is *piecewise affine* if there exists Ω_i as above and

$$(1.11) \quad f|_{\Omega_i} \text{ is affine.}$$

The assumption that Ω_i should have Lipschitz boundary is natural for piecewise affine functions. It can actually be dropped by showing the perturbations we use are always in $W_0^{1,\infty}(\Omega_i)$. For this we only need that the explicit diamond shaped set $\tilde{\Omega}_\varepsilon$ which is defined in the proof of Lemma 3.3 has Lipschitz boundary.

In the following \mathcal{M}^c stands for the convex hull of \mathcal{M} .

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded and open set with Lipschitz boundary. Let L be a compact subset of $\text{Int}(\mathcal{M}^c)$, let $F \in L^\infty(\Omega; \mathbb{R}^{10})$, and suppose that F is piecewise constant and satisfies*

$$(1.12) \quad F(x) \in L \quad \text{a.e.}$$

as well as

$$(1.13) \quad \operatorname{div} D = \operatorname{div} B = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Then there exists a sequence $\{V_j\} = \{(D_j, B_j, P_j, h_j)\} \subset L^\infty(\Omega; \mathbb{R}^{10})$ such that

$$\begin{aligned} \operatorname{div} D_j &= \operatorname{div} B_j = 0 \quad \text{in } \mathcal{D}'(\Omega), \\ V_j &\in \mathcal{M} \quad \text{a.e.}, \\ V_j &\overset{*}{\rightharpoonup} F \quad \text{in } L^\infty \text{ weak}^*. \end{aligned}$$

Of course Theorem 1.1 is useful only if the convex hull of the set \mathcal{M} has nonempty interior. This follows from the following result of Brenier.

THEOREM 1.2 (see [1, Theorem 2]). *The convex hull \mathcal{M}^c satisfies*

$$(1.14) \quad \mathcal{M}^c \supset \{(D, B, P, h) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} : h \geq 1 + |D| + |B| + |P|\},$$

$$(1.15) \quad \mathcal{M}^c \subset \{(D, B, P, h) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} : h \geq \sqrt{1 + |D|^2 + |B|^2 + |P|^2}\}.$$

Proof. To keep this paper self-contained, we provide a short proof for the convenience of the reader. The second inclusion is clear since the function

$$f(D, B, P, h) := \sqrt{1 + |D|^2 + |B|^2 + |P|^2} - h$$

is convex and vanishes in \mathcal{M} .

To prove the first inclusion it suffices to show that for every $s > 0$ the set

$$B_s := \{(D, B, P, s) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} : s \geq 1 + |D| + |B| + |P|\}$$

is contained in \mathcal{M}^c . Now $B_s = \emptyset$ if $s < 1$ and $B_1 = \{1\} \times \{(0, 0, 0)\} \subset \mathcal{M}$. For $s > 1$ the set B_s is convex and compact. We claim that its extreme points are given by

$$\begin{aligned} \operatorname{Ext}(B_s) &= \{(D, B, P, s) : 1 + |D| + |B| + |P| = s, \\ &\quad \text{only one of the vectors } D, B, P \text{ is nonzero}\}. \end{aligned}$$

Let (D, B, P, s) be an extreme point of B_s . Then $1 + |D| + |B| + |P| = s$. Assume $D \neq 0$ and $B \neq 0$. Then $(D + tD/|D|, B - tB/|B|, P, s) \in B_s$ for $|t| < \min(|B|, |D|)$ and thus (D, B, P, s) is not an extreme point of B_s . Similarly one shows that no other two vectors can be simultaneously nonzero.

Since $\operatorname{Ext}(B_s)$ is compact we have $B_s = (\operatorname{Ext}(B_s))^c$. It thus suffices to show that $\operatorname{Ext}(B_s) \subset \mathcal{M}^c$. Consider a point of the form $Y = (D, 0, 0, s)$ with $|D| = s - 1$. This point is a convex combination of $X_\pm := (D, \pm \alpha D, 0, s)$. We have $X_\pm \in \mathcal{M}$ if and only if $1 + (1 + \alpha^2)|D|^2 = s^2$ and such an α exists since $1 + |D|^2 = s^2 - 2(s - 1) < s^2$. Thus $Y \in \mathcal{M}^c$. In the same way one shows that $(0, B, 0, s) \in \mathcal{M}^c$ if $|B| = s - 1$. Finally consider $Y = (0, 0, P, s)$ with $|P| = s - 1$. There exist $d, b \in \mathbb{R}^3$ such that $(d, b, P/|P|)$ is a positively oriented orthonormal basis. Let $D = \sqrt{s - 1}d$, $B = \sqrt{s - 1}b$. Then $d \wedge b = (s - 1)p = P$ and $1 + |D|^2 + |B|^2 + |P|^2 = 1 + (s - 1) + (s - 1) + (s - 1)^2 = s^2$. Hence $(D, B, P, s) \in \mathcal{M}$ and similarly $(-D, -B, P, s) \in \mathcal{M}$. It follows that $(0, 0, P, s) \in \mathcal{M}^c$. \square

Serre [19] has shown the sharper upper bound

$$(1.16) \quad \begin{aligned} \mathcal{M}^c &\subset \{(D, B, P, h) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times (0, \infty) : \\ &\quad h^2 \geq 1 + |D|^2 + |B|^2 + |P|^2 + 2|P - D \wedge B|\}. \end{aligned}$$

Very recently [20] he has proved that

$$(1.17) \quad \mathcal{M}^c = \{(D, B, P, h) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times (0, \infty) : \\ h^2 \geq 1 + |D|^2 + |B|^2 + |P|^2 + 2\sqrt{|P - D \wedge B|^2 + |P \cdot D|^2 + |P \cdot B|^2}\}.$$

The precise form of \mathcal{M}^c is not important for our argument.

2. Notation. For a matrix $A = (A_{ij}) \in \mathbb{M}^{m \times n}$ we denote by A^i the i th column of A and by A_i the i th row of A . We say that a matrix field $V \in L^1(\Omega; \mathbb{M}^{m \times n})$ is divergence free, and we write $\operatorname{Div} V = 0$ in $\mathcal{D}'(\Omega; \mathbb{R}^m)$, if each row of the matrix field V is divergence free in the distributional sense. We denote by \mathcal{M} the six-dimensional manifold in \mathbb{R}^{10} defined as

$$(2.1) \quad \mathcal{M} := \{(D, B, P, h) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} : P = D \wedge B, h = \sqrt{1 + |D|^2 + |B|^2 + |P|^2}\}$$

and by \mathcal{M}^c its convex hull. For the topological interior of \mathcal{M} we write $\operatorname{Int}(\mathcal{M})$. In section 4.2 we use the identification $\mathbb{R}^{10} \simeq \mathbb{R}_D^3 \times \mathbb{R}_B^3 \times \mathbb{R}_P^3 \times \mathbb{R}_h$, and for any $M = (M_1, \dots, M_{10}) \in \mathbb{R}^{10}$, we write

$$M = (M_D, M_B, M_P, M_h) \in \mathbb{R}_D^3 \times \mathbb{R}_B^3 \times \mathbb{R}_P^3 \times \mathbb{R}_h.$$

As usual $W^{1,\infty}(\Omega)$ denotes the Sobolev space of L^∞ functions whose distributional derivative are in L^∞ . By $W_0^{1,\infty}(\Omega)$ we denote the subspace of functions f such that there exist $f_k \in C_c^\infty(\Omega)$ with $(f_k, Df_k) \rightarrow (f, Df)$ a.e. and $\sup_k \|f_k\|_{W^{1,\infty}} < \infty$. If Ω is a bounded open set with Lipschitz boundary, then $W^{1,\infty}(\Omega)$ agrees with the space of functions which have a Lipschitz continuous extension to $\bar{\Omega}$ and the subspace $W_0^{1,\infty}(\Omega)$ consists exactly of Lipschitz functions with $f|_{\partial\Omega} = 0$.

If f is a function on $E \subset \mathbb{R}^n$ we denote by $f\chi_E$ the extension of f by zero to \mathbb{R}^n . If $f \in W_0^{1,\infty}(\Omega)$, then approximation of f by $f_k \in C_c^\infty(\Omega)$ shows that

$$(2.2) \quad f\chi_E \in W^{1,\infty}(\mathbb{R}^n) \quad \text{and} \quad D(f\chi_E) = (Df)\chi_E \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

3. Convex integration for solenoidal fields. As noted in the introduction, extensions of the convex integration method to the div-free case are known. However, for the reader's convenience and because of certain modifications of the existing approaches, we present a self-contained program based on the notion of in-approximation. We will essentially follow [14].

We will work with potentials of divergence-free fields. Therefore we introduce the differential operator $\mathcal{L} : (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m \rightarrow L^\infty(\Omega; \mathbb{M}^{m \times n})$, defined as

$$(\mathcal{L}(G))_{kj} := \sum_{i=1}^n \frac{\partial G_{ij}^k}{\partial x_i}, \quad 1 \leq k \leq m, 1 \leq j \leq n, \quad G = (G^1, \dots, G^m).$$

LEMMA 3.1. *Let $G^k \in W^{1,\infty}(\Omega; \mathbb{M}^{n \times n})$ be matrix fields for $1 \leq k \leq m$ such that the tensor G^k is skew symmetric for every k , i.e., $G_{ij}^k = -G_{ji}^k$. Then the matrix field $\mathcal{L}(G)$ is divergence free.*

Remark 3.2. For $n = 3$, the space of skew symmetric 3×3 matrices $\mathbb{M}_{skw}^{3 \times 3}$ can be identified with \mathbb{R}^3 and the operator \mathcal{L} can alternatively be written as the rowwise curl of a $k \times 3$ matrix. We will, however, not use this fact.

The next result provides the basic construction that allows one to define a divergence-free field whose values lie in a small neighborhood of two values and whose potential can be chosen to be zero on the boundary.

LEMMA 3.3. *Let $A, B \in \mathbb{M}^{m \times n}$ and let $F := \theta A + (1 - \theta)B$ for some $\theta \in (0, 1)$. Assume that $\text{rank}(A - B) \leq n - 1$. Then for each $\delta > 0$, there exists $V \in L^\infty(\Omega; \mathbb{M}^{m \times n})$ such that*

$$\begin{aligned} V &= \mathcal{L}(G) + F \text{ with } G \in (W^{1,\infty}(\Omega; \mathbb{M}_{skw}^{n \times n}))^m \text{ and piecewise linear,} \\ \|G\|_{L^\infty(\Omega)} &< \delta, \\ G|_{\partial\Omega} &= 0, \\ \text{dist}(V, \{A, B\}) &< \delta. \end{aligned}$$

Proof. Without loss of generality we may assume that $(A - B)e_n = 0$, and $F = 0$, so that we can write $A = (1 - \theta)(A - B)$ and $B = -\theta(A - B)$. If not, we can replace A and B by $A - F$ and $B - F$, respectively. We first construct a solution for a special domain Ω_ε and then we will complete the proof by an application of the Vitali covering theorem. Let $\Omega_\varepsilon := (-1, 1)^{n-1} \times (0, \varepsilon)$ and let $\chi : \Omega_\varepsilon \rightarrow \{0, 1\}$ be the characteristic function of the set $(-1, 1)^{n-1} \times (0, \varepsilon\theta)$:

$$\chi(x) = \begin{cases} 1 & \text{if } 0 \leq x_n \leq \varepsilon\theta, \\ 0 & \text{if } \varepsilon\theta < x_n \leq \varepsilon. \end{cases}$$

We then define $U := \chi A + (1 - \chi)B$ and remark that U is divergence free, since $(A - B)e_n = 0$. We seek a potential P of U . For each $k = 1, \dots, m$ and $j = 1, \dots, n$, let

$$\begin{aligned} P_{nj}^k(x) &= \begin{cases} A_{kj}x_n & \text{if } 0 \leq x_n \leq \varepsilon\theta, \\ B_{kj}(x_n - \varepsilon\theta) + \varepsilon\theta A_{kj} & \text{if } \varepsilon\theta < x_n \leq \varepsilon, \end{cases} \\ P_{jn}^k &= -P_{nj}^k, \\ P_{ij}^k &= 0 \text{ otherwise.} \end{aligned}$$

It is readily seen that $U = \mathcal{L}(P)$. Moreover P is piecewise linear and $P = 0$ at $x_n = 0$ and $x_n = \varepsilon$, but P does not vanish on the whole boundary of Ω_ε . In order to find the sought function G , we first remark that for each $k = 1, \dots, m$, the function P_n^k is proportional to $A_k - B_k$, and we compute $\langle P_n^k, A_k - B_k \rangle$:

$$\langle P_n^k, A_k - B_k \rangle = \begin{cases} |A_k - B_k|^2(1 - \theta)x_n & \text{if } 0 \leq x_n \leq \varepsilon\theta, \\ |A_k - B_k|^2\theta(\varepsilon - x_n) & \text{if } \varepsilon\theta < x_n \leq \varepsilon. \end{cases}$$

Note that $\langle P_n^k, A_k - B_k \rangle \geq 0$ in Ω_ε . For each $k = 1, \dots, m$, we introduce the function

$$Q_n^k(x) := -\varepsilon\theta(1 - \theta)(|x_1| + \dots + |x_{n-1}|)(A_k - B_k)$$

and set

$$(3.1) \quad \tilde{P}_n^k := P_n^k + Q_n^k.$$

The function \tilde{P}_n^k is piecewise linear and satisfies $\langle \tilde{P}_n^k, A_k - B_k \rangle \leq 0$ on $\partial\Omega_\varepsilon$. On the other hand $\langle \tilde{P}_n^k, A_k - B_k \rangle > 0$ in a neighborhood of the segment $\{x \in \Omega_\varepsilon : x_1 = \dots = x_{n-1} = 0\}$. Set

$$\tilde{\Omega}_\varepsilon := \{x \in \Omega_\varepsilon : \langle \tilde{P}_n^k, A_k - B_k \rangle > 0\}$$

and define $\tilde{U} := \mathcal{L}(\tilde{P})$, where $\tilde{P} \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m$ is defined by (3.1) and

$$\begin{aligned}\tilde{P}_{jn}^k &= -\tilde{P}_{nj}^k, \\ \tilde{P}_{ij}^k &= 0 \quad \text{otherwise.}\end{aligned}$$

Then

$$\begin{aligned}\tilde{P} &\in (W^{1,\infty}(\tilde{\Omega}_\varepsilon; \mathbb{M}^{n \times n}))^m \text{ is piecewise linear,} \\ \tilde{P}|_{\partial\tilde{\Omega}_\varepsilon} &= 0, \\ \|\tilde{P}\|_{L^\infty(\tilde{\Omega}_\varepsilon)} &< \varepsilon\theta(1-\theta)|A-B|, \\ \text{dist}(\tilde{U}, \{A, B\}) &< \varepsilon\theta(1-\theta)|A-B|.\end{aligned}$$

By the Vitali covering theorem one can exhaust Ω by disjoint scaled copies of $\tilde{\Omega}_\varepsilon$. More precisely, there exist $r_i \in (0, 1)$ and $x_i \in \Omega$ such that the sets $\tilde{\Omega}_\varepsilon^i := x_i + r_i\tilde{\Omega}_\varepsilon$ are mutually disjoint, compactly contained in Ω , and $\text{meas}(\Omega \setminus \cup \tilde{\Omega}_\varepsilon^i) = 0$. Then we define

$$G(x) := \begin{cases} r_i \tilde{P}(r_i^{-1}(x - x_i)) & \text{if } x \in \tilde{\Omega}_\varepsilon^i, \\ 0 & \text{elsewhere.} \end{cases}$$

It follows from (2.2) that $G \in W^{1,\infty}(\Omega)$ and we have $G = 0$ on $\partial\Omega$. We set $V := \mathcal{L}(G)$. By choosing ε sufficiently small, it can be easily checked that V satisfies all the required properties. \square

Next we study the problem of finding a divergence-free field taking values in an open set K and with a prescribed average F . From Lemma 3.3 we know that such problem can be solved provided that $F = \theta A + (1 - \theta)B$ for some $\theta \in (0, 1)$ and $A, B \in K$ with $\text{rank}(A - B) \leq n - 1$. We will see that this procedure can be iterated. More precisely, if $\text{rank}(F - F') \leq n - 1$, and $F' = \theta' A' + (1 - \theta')B'$ for some $\theta' \in (0, 1)$ and $A', B' \in K$, with $\text{rank}(A' - B') \leq n - 1$, then the above problem can be solved also for $\mu F + (1 - \mu)F' \forall \mu \in (0, 1)$. This motivates the following definition.

DEFINITION 3.4. We say that $K \subset \mathbb{M}^{m \times n}$ is stable under lamination (or lamination convex) if $\forall A, B \in K$ such that $\text{rank}(A - B) \leq n - 1$, and all $\theta \in (0, 1)$, one has $\theta A + (1 - \theta)B \in K$. The lamination convex hull K^L is defined as the smallest lamination convex set that contains K .

Remark 3.5. It can be easily checked that the lamination convex hull K^L is obtained by successively adding rank- $(n - 1)$ segments, i.e.,

$$K^L = \bigcup_i K^i,$$

where $K^0 = K$ and

$$\begin{aligned}K^i &:= K^{i-1} \cup \{C : \exists A, B \in K^{i-1}, \theta \in (0, 1) \text{ such that} \\ &C = \theta A + (1 - \theta)B, \text{rank}(A - B) \leq n - 1\}.\end{aligned}$$

Moreover, if K is open, then all the sets K^i are open.

LEMMA 3.6. Suppose that $K \subset \mathbb{M}^{m \times n}$ is open and bounded and that $F \in L^\infty(\Omega; \mathbb{M}^{m \times n})$ is a piecewise constant function which satisfies

$$\begin{aligned}\text{Div} F &= 0 \text{ in } \mathcal{D}'(\Omega; \mathbb{R}^m), \\ F &\in K^L \text{ a.e.}\end{aligned}$$

Then, for each $\delta > 0$, there exists $V_\delta \in L^\infty(\Omega; \mathbb{M}^{m \times n})$ such that

$$\begin{aligned} V_\delta &= \mathcal{L}(G_\delta) + F \text{ with } G_\delta \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m \text{ and piecewise linear,} \\ V_\delta &\in K \quad \text{a.e.,} \\ \|G_\delta\|_{L^\infty(\Omega)} &< \delta, \\ G_\delta|_{\partial\Omega} &= 0. \end{aligned}$$

Proof. We first assume that F is constant. Then $F \in K^i$ for some i . We argue by induction on i . If $i = 1$, then the result holds by Lemma 3.3. Now assume that the result is true $\forall i \leq j$ and let $F \in K^{j+1}$. Then there exist $A, B \in K^j$ such that $\text{rank}(A - B) \leq n - 1$ and $F := \theta A + (1 - \theta)B$ for some $\theta \in (0, 1)$. By Lemma 3.3 there exists a piecewise linear function G such that $\|G\|_{L^\infty(\Omega)} < \delta/2$, $G|_{\partial\Omega} = 0$, and $\text{dist}(\mathcal{L}(G), \{A - F, B - F\}) < \delta$. Since the set K^j is open (see Remark 3.5), for sufficiently small δ , the function $U := \mathcal{L}(G) + F$ satisfies $U \in K^j$ a.e. The latter inclusion implies that U can be written in the form $U = \sum_h \chi_{\Omega_h}(C_h + F)$ with $C_h + F \in K^j$ and with χ_{Ω_h} characteristic functions of disjoint open subsets Ω_h of Ω with Lipschitz boundary and $|\Omega \setminus \bigcup_h \Omega_h| = 0$. We can now apply the induction hypothesis on each subset Ω_h to deduce the existence of functions $G_h \in (W^{1,\infty}(\Omega_h; \mathbb{M}^{n \times n}))^m$ such that

$$\begin{aligned} \mathcal{L}(G_h) + C_h + F &\in K \text{ a.e. in } \Omega_h, \\ \|G_h\|_{L^\infty(\Omega_h)} &< \delta/2, \\ G_h|_{\partial\Omega_h} &= 0. \end{aligned}$$

Finally let $G_\delta(x) := \sum_h \chi_{\Omega_h} G_h + G$. Then $\|G_\delta\|_{L^\infty(\Omega)} < \delta$ and $G_\delta|_{\partial\Omega} = 0$ and by (2.2) we have

$$\mathcal{L}(G_\delta) + F = \sum_h \chi_{\Omega_h}(\mathcal{L}(G_h) + C_h + F) \in K \text{ a.e.}$$

Now let F be piecewise constant. Then $F = \sum_k \chi_{\Omega_k} F_k$ with $F_k \in K^L$. We now use the previous argument in each subdomain Ω_k where F is constant to obtain the existence of piecewise linear functions $G_\delta^k \in (W^{1,\infty}(\Omega_k; \mathbb{M}^{n \times n}))^m$ such that

$$\begin{aligned} \mathcal{L}(G_\delta^k) + F_k &\in K \text{ a.e.,} \\ \|G_\delta^k\|_{L^\infty(\Omega_k)} &< \delta, \\ G_\delta^k|_{\partial\Omega_k} &= 0. \end{aligned}$$

Finally we define $G_\delta := \sum_k \chi_{\Omega_k} G_\delta^k$ and set $V_\delta := \mathcal{L}(G_\delta) + F$. Using again (2.2) we easily deduce the assertion. \square

The next step is to pass from open sets to more general sets $K \subset \mathbb{M}^{m \times n}$. In order to do this we approximate K by open sets \mathcal{U}_i and we construct approximate solutions V_i that satisfy $V_i \in \mathcal{U}_i$. Each of the approximate solutions V_{i+1} is obtained from V_i by an application of Lemma 3.6. This suggests in which sense the sets \mathcal{U}_i have to approximate K .

DEFINITION 3.7. Let $K \subset \mathbb{M}^{m \times n}$. We say that a sequence of nonempty open sets $\{\mathcal{U}_i\} \subset \mathbb{M}^{m \times n}$ is an in-approximation of K if the following three conditions hold:

1. $\mathcal{U}_i \subset \mathcal{U}_{i+1}^L$;
2. the sets \mathcal{U}_i are uniformly bounded;
3. if a sequence $F_i \in \mathcal{U}_i$ converges to F as $i \rightarrow \infty$, then $F \in K$.

The name “in-approximation” was introduced by Gromov [10]. Note that a necessary condition for K to admit an in-approximation is that the set $\text{Int}(K^L)$ is nonempty. Note also that the notion of in-approximation is related to a notion of convexity. In this section we use lamination convexity with respect to the cone of matrices of rank (at most) $n - 1$ because Lemma 3.3 holds only if $\text{rank}(A - B) \leq n - 1$. In the next section we will prove a similar lemma, but without any restriction. Thus in that section the natural cone is the whole space (in that case \mathbb{R}^{10}), and in condition 1 in the in-approximation we will use the ordinary convex hull.

We are now ready to state the main result of this section.

THEOREM 3.8. *Assume that K admits an in-approximation by open sets \mathcal{U}_i and let $F \in \mathcal{U}_1$. Then, for each $\delta > 0$, there exists $V_\delta \in L^\infty(\Omega; \mathbb{M}^{m \times n})$ such that*

$$(3.2) \quad V_\delta = \mathcal{L}(H_\delta) + F \text{ with } H_\delta \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m,$$

$$(3.3) \quad V_\delta \in K \quad \text{a.e.},$$

$$(3.4) \quad \|H_\delta\|_{L^\infty(\Omega)} < \delta,$$

$$(3.5) \quad H_\delta|_{\partial\Omega} = 0.$$

Proof. We construct a sequence of piecewise constant divergence-free maps V_i such that

$$(3.6) \quad V_i = \mathcal{L}(H_i) + F \text{ with } H_i \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m,$$

$$V_i \in \mathcal{U}_i \quad \text{a.e.},$$

$$\|H_{i+1} - H_i\|_{L^\infty(\Omega)} < \delta_{i+1},$$

$$H_i|_{\partial\Omega} = 0.$$

To start, set $H_1 := 0$ and $V_1 := F$. Since $F \in \mathcal{U}_2^L$, we can apply Lemma 3.6 to deduce the existence of a function V_2 such that

$$V_2 = \mathcal{L}(G_2) + F \text{ with } G_2 \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m \text{ and piecewise linear,}$$

$$V_2 \in \mathcal{U}_2 \quad \text{a.e.},$$

$$\|G_2\|_{L^\infty(\Omega)} < \delta_2,$$

$$G_2|_{\partial\Omega} = 0$$

with $\delta_2 = \delta$. We then define $H_2 = G_2$. To construct V_{i+1} and δ_{i+1} from V_i and δ_i , we proceed as follows. Let

$$\Omega_i := \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/2^i\}.$$

Let ϱ be a standard smooth convolution kernel in \mathbb{R}^n , i.e., $\rho \geq 0$, $\int \rho = 1$, $\text{Spt } \rho \subset \{|x| < 1\}$, and let $\varrho_{\varepsilon_i}(x) := \varepsilon_i^{-n} \varrho(x/\varepsilon_i)$. We choose $\varepsilon_i \in (0, 2^{-i})$ so that

$$(3.7) \quad \|\varrho_{\varepsilon_i} * \mathcal{L}(H_i) - \mathcal{L}(H_i)\|_{L^1(\Omega_i)} < \frac{1}{2^i},$$

where the convolution acts on each entry of the matrix field $\mathcal{L}(H_i)$. Now let

$$(3.8) \quad \delta_{i+1} = \delta_i \varepsilon_i$$

and use Lemma 3.6 to construct a function $G_{i+1} \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m$ such that

$$\begin{aligned}\mathcal{L}(G_{i+1}) + V_i &\in \mathcal{U}_{i+1} \quad \text{a.e.}, \\ \|G_{i+1}\|_{L^\infty(\Omega)} &< \delta_{i+1}.\end{aligned}$$

Next we set $H_{i+1} := \sum_{j=2}^{i+1} G_j$ and define V_{i+1} according to (3.6), so that

$$V_{i+1} = \mathcal{L}(G_{i+1}) + V_i.$$

Since $\sum_{i=2}^\infty \delta_i < \delta/2$ and, for $i > j$,

$$(3.9) \quad \|H_i - H_j\|_{L^\infty(\Omega)} \leq \sum_{k=j+1}^i \|G_k\|_{L^\infty(\Omega)},$$

we find that $H_i \rightarrow H_\infty$ uniformly. Moreover, since by construction the sequence $\{H_i\}$ is uniformly bounded in $W^{1,\infty}(\Omega)$, we have that $H_i \xrightarrow{*} H_\infty$ in $W^{1,\infty}$ weak*. In particular

$$\mathcal{L}(H_i) \xrightarrow{*} \mathcal{L}(H_\infty) \text{ in } L^\infty \text{ weak } *.$$

Taking $H_\delta = H_\infty$ and $V_\delta := \mathcal{L}(H_\delta) + F$, we see that conditions (3.2), (3.4), and (3.5) hold. We are left to show that $V_\delta \in K$ a.e. To this end, we will prove the strong convergence of $\mathcal{L}(H_i)$ to $\mathcal{L}(H_\infty)$ in L^1 . Indeed, since

$$\int_{\Omega} (\mathcal{L}(\Phi)(y))_{kj} \varrho(x-y) dy = - \sum_{i=1}^n \int_{\Omega} \Phi_{ij}^k(y) \frac{\partial \varrho}{\partial x_i}(x-y) dy \quad \forall \Phi \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m,$$

and since $\|\nabla \varrho_{\varepsilon_i}\|_{L^1} < C/\varepsilon_i$, we deduce from (3.8) and (3.9)

$$\begin{aligned}(3.10) \quad \|\varrho_{\varepsilon_i} * (\mathcal{L}(H_i) - \mathcal{L}(H_\infty))\|_{L^1(\Omega_i)} &\leq \frac{C}{\varepsilon_i} \|H_i - H_\infty\|_{L^\infty(\Omega)} \\ &\leq \frac{C}{\varepsilon_i} \sum_{k=i+1}^\infty \delta_k \\ &\leq 2 \frac{C}{\varepsilon_i} \delta_{i+1} \\ &\leq C' \delta_i.\end{aligned}$$

Combining (3.7) and (3.10) we get

$$\begin{aligned}\|\mathcal{L}(H_i) - \mathcal{L}(H_\infty)\|_{L^1(\Omega)} &\leq C' \delta_i + 2^{-i} + \|\varrho_{\varepsilon_i} * \mathcal{L}(H_\infty) - \mathcal{L}(H_\infty)\|_{L^1(\Omega_i)} \\ &\quad + \|\mathcal{L}(H_i) - \mathcal{L}(H_\infty)\|_{L^1(\Omega \setminus \Omega_i)}.\end{aligned}$$

Since $\mathcal{L}(H_i)$ and $\mathcal{L}(H_\infty)$ are bounded, we obtain $\mathcal{L}(H_i) \rightarrow \mathcal{L}(H_\infty)$ in $L^1(\Omega)$ and thus $V_i \rightarrow V_\delta$ in $L^1(\Omega)$. Therefore there exists a subsequence V_{i_j} such that

$$V_{i_j} \rightarrow V_\delta \quad \text{a.e.}$$

It follows from the definition of in-approximation that

$$V_\delta \in K \quad \text{a.e.} \quad \square$$

4. Applications of the convex integration results to the study of the Born–Infeld equations.

4.1. Approach by Young measures. We formulate problem (1.8)–(1.9) in the language of \mathcal{A} -convexity (see, e.g., [8], [24]). Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain and let \mathcal{M} be defined by (2.1). Let $A^{(1)}, A^{(2)}, A^{(3)} \in \mathbb{M}^{2 \times 10}$ be defined as follows:

$$\begin{aligned} A^{(1)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ A^{(2)} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ A^{(3)} &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

We introduce the operators

$$\begin{aligned} \mathcal{A}(V) &:= \sum_{i=1}^3 A^{(i)} \frac{\partial V}{\partial x_i}, \quad V : \Omega \rightarrow \mathbb{R}^{10}, \\ \mathbb{A}(w) &:= \sum_{i=1}^3 A^{(i)} w_i \in \text{Lin}(\mathbb{R}^{10}; \mathbb{R}^2), \quad w \in \mathbb{R}^3, \end{aligned}$$

where $\text{Lin}(\mathbb{R}^{10}; \mathbb{R}^2)$ denotes the space of linear operators from \mathbb{R}^{10} to \mathbb{R}^2 . The operator \mathcal{A} satisfies the *constant rank* property, i.e.,

$$\text{rank} \mathbb{A}(w) = 2 \quad \forall w \in \mathbb{S}^2,$$

where \mathbb{S}^2 is the unit sphere in \mathbb{R}^3 . Moreover

$$\ker \mathbb{A}(w) = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^4 : \alpha \perp w, \beta \perp w\} = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^4.$$

Therefore the *characteristic cone* Λ is all of \mathbb{R}^{10} . Indeed

$$\begin{aligned} \Lambda &:= \cup_{w \in \mathbb{S}^2} \ker \mathbb{A}(w) \\ &= \{(\alpha, \beta) \in \mathbb{R}^3 \times \mathbb{R}^3 : \exists \xi \in \mathbb{R}^3 \text{ such that } \xi \perp \alpha, \xi \perp \beta\} \times \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^4. \end{aligned}$$

Thus Λ -convexity reduces to standard convexity. In terms of the constant rank operator \mathcal{A} our problem reads as

$$(4.1) \quad \mathcal{A}(V_j) = 0 \quad \text{in } \mathcal{D}'(\Omega),$$

$$(4.2) \quad V_j \in \mathcal{M} \quad \text{a.e. in } \Omega.$$

One can also consider the approximate version of (4.1), where the differential constraint on the sequence $\{V_j\}$ is replaced by the weaker condition

$$(4.3) \quad \mathcal{A}(V_j) \rightarrow 0 \quad \text{strongly in } W^{-1,p'}(\Omega).$$

Theorems 4.1 and 4.3 and their corollaries are a special case of more general results contained in [8], where more general constant rank operators are considered. Let us also mention that in the gradient case, i.e., when the operator \mathcal{A} is the curl operator, such results were first established by Kinderlehrer and Pedregal [11].

THEOREM 4.1. *Let $1 \leq p < +\infty$. Suppose that the sequence $\{V_j\}$ generates the Young measure $\{\nu_x\}_{x \in \Omega}$ and let $V_j \rightharpoonup V$ in $L^p(\Omega; \mathbb{R}^{10})$. If $\{V_j\}$ satisfies (4.1), or its approximate version (4.3), then*

$$\langle \nu_x, id \rangle = V(x) \in \ker \mathcal{A},$$

$$\int_{\Omega} \int_{\mathbb{R}^{10}} |M|^p d\nu_x(M) < \infty.$$

If in addition the sequence $\{V_j\}$ is uniformly bounded in $L^\infty(\Omega; \mathbb{R}^{10})$ and (4.2) holds, then

$$(4.4) \quad \text{supp } \nu_x \subset \mathcal{M} \text{ for a.e. } x \in \Omega.$$

COROLLARY 4.2. Under the assumptions of Theorem 4.1, if (4.4) holds, then

$$V(x) \in \mathcal{M}^c \text{ for a.e. } x \in \Omega.$$

THEOREM 4.3. Let $1 \leq p < +\infty$, and let $\{\nu_x\}_{x \in \Omega}$ be a weakly measurable family of probability measures on \mathbb{R}^{10} . Suppose that

$$\langle \nu_x, id \rangle \in \ker \mathcal{A},$$

$$\int_{\Omega} \int_{\mathbb{R}^{10}} |M|^p d\nu_x(M) < \infty.$$

Then there exists a sequence $\{V_j\} \subset L^p(\Omega; \mathbb{R}^{10})$ satisfying (4.1) that generates $\{\nu_x\}$.

COROLLARY 4.4. Let $V \in L^p(\Omega; \mathbb{R}^{10})$. Suppose that $\mathcal{A}(V) = 0$ and $V \in \mathcal{M}^c$ a.e. Then there exists a sequence $\{V_j\} \subset L^p(\Omega; \mathbb{R}^{10})$ satisfying (4.1) such that

$$\text{dist}(V_j, \mathcal{M}) \rightarrow 0 \text{ in } L^p(\Omega) \text{ and } V_j \rightharpoonup V \text{ in } L^p(\Omega; \mathbb{R}^{10}).$$

Remark 4.5. By suitably projecting the sequence $\{V_j\}$ provided by Corollary 4.4 onto \mathcal{M} , one can obtain a sequence $\{\tilde{V}_j\} \subset L^p(\Omega; \mathbb{R}^{10})$ satisfying (4.3) such that

$$\tilde{V}_j \in \mathcal{M} \text{ a.e. and } \tilde{V}_j \rightharpoonup V \text{ in } L^p(\Omega).$$

4.2. Approach by convex integration. We now use the convex integration approach developed in section 3 to find maps which satisfy the constraints (4.1) and (4.2) exactly and have a prescribed average in the interior of the convex hull \mathcal{M}^c . Then Theorem 1.1 will follow easily by partitioning Ω into small subdomains and applying the result to each subdomain.

As above we write

$$M = (M_D, M_B, M_P, M_h) \in \mathbb{R}_D^3 \times \mathbb{R}_B^3 \times \mathbb{R}_P^3 \times \mathbb{R}_h.$$

We look for maps

$$V : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^{10}$$

which satisfy the constraints $\text{div} V_D = \text{div} V_B = 0$. Since we take the divergence of a matrix rowwise this constraint can be written in the compact form

$$\text{div} \begin{pmatrix} V_D^T \\ V_P^T \end{pmatrix} = 0, \quad \text{where } \begin{pmatrix} V_D^T \\ V_B^T \end{pmatrix} \in \mathbb{M}^{2 \times 3}.$$

We first state the counterpart of Lemma 3.3 in the present setting. The operator \mathcal{L} is the same as in the previous section. As we work with $n = 3$ we could identify \mathcal{L}

with the rowwise curl operator of a matrix, but we refrain from doing so to keep the notation as close as possible to the previous section.

LEMMA 4.6. *Let $M, N \in \mathbb{R}^{10}$ and let $F := \theta M + (1 - \theta)N$ for some $\theta \in (0, 1)$. Then for each $\delta > 0$, there exists $V \in L^\infty(\Omega; \mathbb{R}^{10})$ such that*

$$(4.5) \quad \begin{pmatrix} V_D^T \\ V_B^T \end{pmatrix} = \begin{pmatrix} F_D^T \\ F_B^T \end{pmatrix} + \mathcal{L}(G) \text{ with } G \in (W^{1,\infty}(\Omega; \mathbb{M}_{skw}^{3 \times 3}))^2 \text{ and piecewise linear,}$$

$$(4.6) \quad \|G\|_{L^\infty(\Omega)} < \delta,$$

$$(4.7) \quad G|_{\partial\Omega} = 0,$$

$$(4.8) \quad \text{dist}(V, \{M, N\}) < \delta,$$

$$(4.9) \quad \int_\Omega V \, dx = F|\Omega|.$$

Proof. By scaling we may assume without loss of generality $|\Omega| = 1$. We apply Lemma 3.3 with

$$(4.10) \quad A = \begin{pmatrix} M_D^T \\ M_B^T \end{pmatrix}, \quad B = \begin{pmatrix} N_D^T \\ N_B^T \end{pmatrix}$$

and with δ' instead of δ . Note that $A, B \in \mathbb{M}^{2 \times 3}$ and hence

$$\text{rank}(A - B) \leq 2 = n - 1.$$

It follows that there exists a piecewise linear $G \in (W_0^{1,\infty}(\Omega; \mathbb{M}_{skw}^{3 \times 3}))^2$ such that (4.5) and (4.7) hold and

$$(4.11) \quad \|G\|_{L^\infty(\Omega)} < \delta', \quad \text{dist} \left(\begin{pmatrix} V_D^T \\ V_B^T \end{pmatrix}, \{A, B\} \right) < \delta'.$$

Moreover

$$\int_\Omega (V_D, V_B) = (F_D, F_B)$$

since $G = 0$ on $\partial\Omega$.

It remains only to define V_P and V_h . Since no differential constraint is imposed on these variables this is easy. We set

$$\Omega_A := \left\{ x \in \Omega : \left| \begin{pmatrix} V_D^T \\ V_B^T \end{pmatrix} - A \right| < \left| \begin{pmatrix} V_D^T \\ V_B^T \end{pmatrix} - B \right| \right\}.$$

Denote by χ_A be the characteristic function of Ω_A and define

$$\eta := \int_\Omega \chi_A = |\Omega_A|.$$

We set

$$(V_P, V_h) = \chi_A(M_P, M_h) + (1 - \chi_A)(N_P, N_h) + (\theta - \eta)(M_P - N_P, M_h - N_h).$$

Then by the definition of η (recall that $|\Omega| = 1$)

$$\int_\Omega (V_P, V_h) \, dx = (F_P, F_h).$$

Moreover the definition of Ω_A and the second inequality in (4.11) imply that

$$(4.12) \quad \text{dist}(V, \{(M, N)\}) \leq \delta' + |\theta - \eta| |(M_P - N_P, M_h - N_h)|.$$

To estimate $\eta - \theta$ we note that

$$\begin{aligned} \begin{pmatrix} F_D^T \\ F_B^T \end{pmatrix} - \eta A - (1 - \eta)B &= \int_{\Omega} \begin{pmatrix} V_D^T \\ V_B^T \end{pmatrix} dx - \eta A - (1 - \eta)B \\ &= \int_{\Omega_A} \begin{pmatrix} V_D^T \\ V_B^T \end{pmatrix} - A \, dx + \int_{\Omega \setminus \Omega_A} \begin{pmatrix} V_D^T \\ V_B^T \end{pmatrix} - B \, dx. \end{aligned}$$

Taking the norm on both sides and using the definition of Ω_A and (4.11) we see that $|(\theta - \eta)(A - B)| \leq \delta'$. Now take

$$\delta' := \frac{1}{2} \delta \frac{|A - B|}{|M - N|} \leq \frac{1}{2} \delta.$$

Then (4.8) follows from (4.12). \square

Now we introduce the appropriate definition of in-approximation for the Born-Infeld set \mathcal{M} defined in (2.1). Note that while in Lemma 3.3 we had the constraint $\text{rank}(A - B) \leq n - 1$, in Lemma 4.6 there is no constraint at all on the matrices M and N . Thus the lamination convex hull introduced in the previous section is replaced by the ordinary convex hull and the in-approximation is defined using the convex hull. Note that by Caratheodory's theorem the convex hull of any set $E \subset \mathbb{R}^{10}$ satisfies $E^c = \bigcup_{i=0}^{10} E^i$, where $E^0 = A$ and E^{i+1} is inductively defined as the set all convex combinations $\theta A + (1 - \theta)B$, with $A, B \in E^i$.

DEFINITION 4.7. *We say that a sequence of nonempty open sets $\{\mathcal{U}_i\} \subset \mathbb{R}^{10}$ is an in-approximation of \mathcal{M} if the following three conditions hold:*

1. $\mathcal{U}_i \subset \mathcal{U}_{i+1}^c$;
2. the sets \mathcal{U}_i are uniformly bounded;
3. if a sequence F_i converges to F as $i \rightarrow \infty$ and $F_i \in \mathcal{U}_i$ for each i , then $F \in \mathcal{M}$.

Regarding the existence of in-approximations with respect to ordinary convexity we have the following abstract result.

LEMMA 4.8. *Let $M \subset \mathbb{R}^d$, assume that $\text{Int} M^c \neq \emptyset$, and let $L \subset \text{Int} M^c$ be compact. Then there exist $R > 0$ and open sets \mathcal{U}_i such that*

1. $L \subset \mathcal{U}_1$,
2. $\mathcal{U}_i \subset \mathcal{U}_{i+1}^c \, \forall i \geq 1$,
3. $\mathcal{U}_i \subset B(0, R) \, \forall i \geq 1$,
4. $\mathcal{U}_i \subset B_{1/i}(M) \, \forall i \geq 2$.

In particular the sets \mathcal{U}_i are an in-approximation of M .

Note that $\mathcal{U}_i \subset B_{1/i}(M)$ if and only if $\text{dist}(p, M) < \frac{1}{i} \, \forall p \in \mathcal{U}_i$.

To prove Lemma 4.8 we will inductively use the following elementary result for finite sets.

LEMMA 4.9. *Let $E \subset \mathbb{R}^d$ be a finite set, let F be a finite set with*

$$(4.13) \quad F \subset \text{Int} E^c,$$

and let $\varepsilon > 0$. Then there exists a finite set F' such that

$$(4.14) \quad F \subset \text{Int}(F')^c, \quad F' \subset \text{Int} E^c, \quad F' \subset B_\varepsilon(E).$$

Proof. Step 1. Assume that $F = \{0\}$. By assumption there exists $\eta > 0$ such that

$$B_\eta(0) \subset E^c.$$

Let $\delta \in (0, 1)$ and set $F' = (1 - \delta)E$. Then $(F')^c \supset B_{(1-\delta)\eta}(0)$ and hence $0 \in \text{Int}(F')^c$. Since E^c is convex we also have for every $p \in E$ the inclusion $\delta B_\eta(0) + (1 - \delta)p \subset E^c$. Thus $F' \subset \text{Int}E^c$.

Finally if $\delta < \varepsilon / \max\{|p| : p \in E\}$ we have $F' \subset B_\varepsilon(E)$.

Step 2. General finite F . Let

$$\delta < \frac{\varepsilon}{\max\{|q - p| : q \in F, p \in E\}}.$$

For $q \in F$ define

$$F'_q := q + (1 - \delta)(-q + E).$$

By Step 1

$$q \in \text{Int}(F'_q)^c, \quad F'_q \subset \text{Int}E^c, \quad F'_q \subset B_\varepsilon(E).$$

Thus $F' := \bigcup_{q \in F} F'_q$ has the desired properties. \square

Proof of Lemma 4.8. For each $q \in L$ there exists a $\delta(q) > 0$ such that the cube $q + (-3\delta, 3\delta)^d$ is contained in M^c . Since L is compact there exist $q_1, \dots, q_m \in L$ such that

$$L \subset \bigcup_{i=1}^m q_i + (-\delta_i, \delta_i)^d \quad \text{and} \quad q_i + (-3\delta_i, 3\delta_i)^d \subset M^c.$$

Let F_1 be the set of all the corner points of all the cubes $q_i + [-\delta_i, \delta_i]^d$. Then

$$(4.15) \quad L \subset F_1^c.$$

We now show that there exists a finite set $E \subset M$ such that $F_1 \subset \text{Int}E^c$. Indeed, let G_1 denote the set of all the corner points of the cubes $q_i + [-2\delta_i, 2\delta_i]^d$. By Caratheodory's theorem each point in G_1 is a convex combination of at most $d + 1$ points in M . Thus there exists a finite set $E \subset M$ such that $G_1 \subset E^c$. This implies that

$$F_1 \subset \text{Int}G_1^c \subset \text{Int}E^c.$$

Set

$$R := \max\{|p| : p \in E\}.$$

Inductive application of Lemma 4.9 yields finite sets F_i with $F_i \subset \text{Int}E^c$ for all $i \geq 1$ and

$$(4.16) \quad F_i \subset \text{Int}F_{i+1}^c \quad \forall i \geq 1, \quad F_i \subset B_{\frac{1}{i}}(E) \quad \forall i \geq 2.$$

Moreover the condition $F_i \subset \text{Int}E^c$ implies that

$$F_i \subset B(0, R) \quad \forall i \geq 1.$$

Now define

$$\begin{aligned}\mathcal{U}_1 &:= \text{Int} F_2^c, \\ \mathcal{U}_i &:= \text{Int} F_{i+1}^c \cap B_{\frac{1}{i}}(E) \quad \forall i \geq 2.\end{aligned}$$

Then the sets \mathcal{U}_i are open and $\mathcal{U}_i \subset B(0, R)$ since $F_{i+1} \subset B(0, R)$. Moreover by (4.16) we have $F_i \subset \mathcal{U}_i \forall i \geq 1$ and thus

$$\mathcal{U}_i \subset F_{i+1}^c \subset \mathcal{U}_{i+1}^c.$$

Since \mathcal{U}_1 is convex the inclusion $F_1 \subset \mathcal{U}_1$ and (4.15) imply that $\mathcal{U}_1 \supset F_1^c \supset L$. This finishes the proof of Lemma 4.8. \square

For future reference we also note the following observation.

LEMMA 4.10. *For every $\delta > 0$ and every $R > 0$ there exists an $\eta > 0$ such that the estimates*

$$(4.17) \quad |(D, B, P, h)| \leq R \quad \text{and} \quad \text{dist}((D, B, P, h), \mathcal{M}) < \eta$$

imply that

$$(4.18) \quad |P - D \wedge B| < \delta \quad \text{and} \quad |h - \sqrt{1 + |D|^2 + |B|^2 + |P|^2}| < \delta.$$

Proof. This just follows from the continuity of the functions involved and compactness. Indeed, if the assertion is false, then there exist $R_0 > 0$ and $\delta_0 > 0$ such that for each $k \in \mathbb{N}$, $k \geq 1$ there exist $(D_k, B_k, P_k, h_k) \in \bar{B}(0, R_0)$ such that

$$\text{dist}((D_k, B_k, P_k, h_k), \mathcal{M}) \leq \frac{1}{k}$$

and

$$(4.19) \quad |P_k - D_k \wedge B_k| + |h_k - \sqrt{1 + |D_k|^2 + |B_k|^2 + |P_k|^2}| \geq \delta_0.$$

There exists a subsequence such that $(D_{k_j}, B_{k_j}, P_{k_j}, h_{k_j}) \rightarrow (D, B, P, h)$ and $(D, B, P, h) \in \mathcal{M}$. Passage to the limit in (4.19) along this subsequence yields

$$(4.20) \quad |P - D \wedge P| + |h - \sqrt{1 + |D|^2 + |B|^2 + |P|^2}| \geq \delta_0,$$

but this contradicts the definition of \mathcal{M} . \square

We can now construct solutions of the problem $V \in \mathcal{M}$ and $\text{div} V_D = \text{div} V_B = 0$ in complete analogy with the argument in the previous section.

LEMMA 4.11. *Let $U \subset \mathbb{R}^d$ be open and bounded. Suppose that $F \in L^\infty(\Omega; \mathbb{R}^{10})$ is a piecewise constant function which satisfies*

$$\begin{aligned}\text{div} F_D &= \text{div} F_B = 0 \quad \text{in } \mathcal{D}'(\Omega), \\ F &\in U^c \quad \text{a.e.}\end{aligned}$$

Then, for each $\delta > 0$, there exists $V_\delta \in L^\infty(\Omega; \mathbb{R}^{10})$ such that

$$(4.21) \quad \begin{pmatrix} (V_\delta)_D^T \\ (V_\delta)_B^T \end{pmatrix} = F + \mathcal{L}(G) \quad \text{with } G \in (W^{1,\infty}(\Omega; \mathbb{M}_{skw}^{3 \times 3}))^2 \text{ and piecewise linear,}$$

$$(4.22) \quad \|G\|_{L^\infty(\Omega)} < \delta,$$

$$(4.23) \quad G|_{\partial\Omega} = 0,$$

$$(4.24) \quad V_\delta \in U \quad \text{a.e.,}$$

$$(4.25) \quad \int_\Omega V_\delta \, dx = \int_\Omega F \, dx.$$

Proof. Lemma 4.11 follows by induction from Lemma 4.6 exactly in the same way as Lemma 3.6 was deduced from Lemma 3.3. \square

THEOREM 4.12. *Let L be a compact subset of $\text{Int}\mathcal{M}^c$. Then there exists an $R > 0$ such that for all $F \in L$ there exists $V \in L^\infty(\Omega; \mathbb{R}^{10})$ such that*

$$(4.26) \quad \begin{pmatrix} V_D^T \\ V_B^T \end{pmatrix} = \begin{pmatrix} F_D^T \\ F_B^T \end{pmatrix} + \mathcal{L}(H) \text{ with } H \in (W^{1,\infty}(\Omega; \mathbb{M}_{skw}^{3 \times 3}))^2,$$

$$(4.27) \quad \|H\|_{L^\infty(\Omega)} < 1,$$

$$(4.28) \quad H|_{\partial\Omega} = 0,$$

$$(4.29) \quad V \in \mathcal{M} \text{ a.e. and } \|V\|_{L^\infty} \leq R,$$

$$(4.30) \quad \int_\Omega V \, dx = F.$$

Proof. By Lemma 4.8 there exists an in-approximation \mathcal{U}_i with $L \subset \mathcal{U}_1$. Arguing exactly as in the proof of Theorem 3.8 and using now Lemma 4.11 instead of Lemma 3.6 we can inductively define δ_i , H_i , and ε_i such that

$$\begin{aligned} \begin{pmatrix} (V_i)_D^T \\ (V_i)_B^T \end{pmatrix} &= \begin{pmatrix} F_D^T \\ F_B^T \end{pmatrix} + \mathcal{L}(H_i), \\ H_i &\in (W^{1,\infty}(\Omega, \mathbb{M}_{skw}^{3 \times 3}))^2 \text{ and piecewise linear,} \\ H_i|_{\partial\Omega} &= 0, \\ V_i &\in \mathcal{U}_i \text{ a.e.,} \\ \int_\Omega V_i \, dx &= F |\Omega|, \\ \|\rho_{\varepsilon_i} * \mathcal{L}(H_i) - \mathcal{L}(H_i)\|_{L^1(\Omega_i)} &< \frac{1}{2^i}, \\ \delta_{i+1} &= \delta_i \varepsilon_i, \\ \|H_{i+1} - H_i\|_{L^\infty(\Omega)} &< \delta_{i+1}. \end{aligned}$$

Then we get $\mathcal{L}(H_i) \xrightarrow{*} \mathcal{L}(H)$ in $L^\infty(\Omega)$, $\|H\| < \delta \leq 1$, and $\mathcal{L}(H_i) \rightarrow \mathcal{L}(H)$ in $L^1(\Omega)$. This implies that

$$(4.31) \quad (V_i)_D \rightarrow V_D, \quad (V_i)_B \rightarrow V_B \quad \text{in } L^p(\Omega) \, \forall p < \infty.$$

Thus

$$(V_i)_D \wedge (V_i)_B \rightarrow V_D \wedge V_B \quad \text{in } L^p(\Omega) \, \forall p < \infty.$$

By the construction of the in-approximation we have $\|\text{dist}(V_i, \mathcal{M})\|_{L^\infty} \leq \frac{1}{i}$ and $\|V_i\|_{L^\infty} \leq R$ (where R depends only on L). Thus Lemma 4.10 implies that

$$\|(V_i)_P - (V_i)_D \wedge (V_i)_B\|_{L^\infty} \rightarrow 0$$

and thus

$$(V_i)_P \rightarrow V_D \wedge V_B \quad \text{in } L^p(\Omega)$$

$\forall p < \infty$. Similarly Lemma 4.10 implies that

$$\left\| (V_i)_h - \sqrt{1 + |(V_i)_D|^2 + |(V_i)_B|^2 + |(V_i)_P|^2} \right\|_{L^\infty(\Omega)} \rightarrow 0$$

and therefore $(V_i)_h \rightarrow V_h$ strongly in $L^p(\Omega)$ and $V \in \mathcal{M}$ a.e. Since $|V_i| \leq R$ a.e it follows also that $|V| \leq R$ a.e. \square

Proof of Theorem 1.1. Let $L \subset \text{Int}\mathcal{M}^c$ be compact and suppose that $F \in L^\infty(\Omega, \mathbb{R}^{10})$ is piecewise constant with $F(x) \in L$ a.e. Assume furthermore that $\text{div}B = \text{div}D = 0$ in the sense of distributions where $F = (D, B, P, h)$.

To construct the approximation V_j we may assume that

$$(4.32) \quad \text{diam } \Omega_i \leq \frac{1}{j}$$

$\forall i$ since otherwise we can always subdivide all the sets Ω_i with larger diameter until this condition is satisfied.

Now we apply Theorem 4.12 to Ω_i and F_i and we obtain a function $V_i^j : \Omega_i \rightarrow \mathbb{R}^{10}$ and a potential $H_i^j : \Omega_i \rightarrow (\mathbb{M}_{skw}^{3 \times 3})^2$. We extend H_i^j and $V_i^j - F_i$ by zero outside Ω_i . Using again (2.2) we see that these extensions satisfy

$$\chi_{\Omega_i} \left(\begin{pmatrix} (V_i^j)_D^T \\ (V_i^j)_B^T \end{pmatrix} - \begin{pmatrix} F_D^T \\ F_B^T \end{pmatrix} \right) = \mathcal{L}(\chi_{\Omega_i} H_i^j)$$

and thus $\text{div} \chi_{\Omega_i} (V_i^j - F_i)_B = \text{div} \chi_{\Omega_i} (V_i^j - F_i)_D = 0$ in the sense of distributions in \mathbb{R}^3 . Finally we set

$$(4.33) \quad V^j = F + \sum_i \chi_{\Omega_i} (V_i^j - F_i).$$

Then $\text{div}B^j = \text{div}D^j = 0$, where $V^j = (D^j, B^j, P^j, h^j)$. Moreover $V^j = V_i^j$ in Ω_i and hence $V^j \in \mathcal{M}$ a.e.

It remains to show that $V^j \xrightarrow{*} F$ in $L^\infty(\Omega)$. First consider Lipschitz continuous test functions φ and let x_i be a point in Ω_i . Then

$$(4.34) \quad \int_{\Omega} (V^j - F) \varphi \, dx = \sum_i \int_{\Omega_i} (V_i^j - F_i) \varphi \, dx = \sum_i \int_{\Omega_i} (V_i^j(x) - F_i) (\varphi(x) - \varphi(x_i)) \, dx,$$

where we used that $V_i^j - F_i$ has zero average in Ω_i . Now $|V_i^j - F_i| \leq 2R$ and $|\varphi(x) - \varphi(x_i)| \leq \text{Lip } \varphi \, \text{diam } \Omega_i \leq \frac{1}{j} \text{Lip } \varphi$. Thus

$$(4.35) \quad \int_{\Omega} (V^j - F) \varphi \, dx \rightarrow 0$$

for all Lipschitz continuous φ . Since these functions are dense in L^1 and $\|V^j - F\|_{L^\infty} \leq 2R$ the convergence (4.35) holds for all $\varphi \in L^1$. This finishes the proof. \square

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